

Casimir effect for a dilute dielectric ball at finite temperature

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Abstract

The Casimir effect at finite temperature is investigated for a dilute dielectric ball; i.e., the relevant internal and free energies are calculated. The starting point in this study is a rigorous general expression for the internal energy of a system of noninteracting oscillators in terms of the sum over the Matsubara frequencies. Summation over the angular momentum values is accomplished in a closed form by making use of the addition theorem for the relevant Bessel functions. For removing the divergences the renormalization procedure is applied that has been developed in the calculation of the corresponding Casimir energy at zero temperature. The behavior of the thermodynamic characteristics in the low and high temperature limits is investigated.

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I. INTRODUCTION

The calculation of the vacuum electromagnetic energy of a dielectric ball has a rather long history [1–3]. Only recently was the final result obtained for a dilute dielectric ball at zero temperature. Remarkably, this Casimir energy was calculated by different methods: (i) by summing up the van der Waals forces between the individual molecules inside the compact ball [4]; (ii) in the framework of a special perturbation theory, in which the dielectric ball was treated as a perturbation when considering the electromagnetic field in unbounded empty space [5]; (iii) by making use of the Green's functions of the quantized Maxwell field with an explicit account of the so called contact terms [6–8], on the stage of the numerical calculations the uniform asymptotic expansion for the Bessel functions and the zeta regularization technique being applied; (iv) by the mode summation method with the use of the addition theorem for the Bessel functions [9]. In calculations without using the uniform asymptotic expansions for the Bessel functions [4,5,9] the exact (in the Δn^2 approximation) result for the Casimir energy under consideration was obtained. The general structure of the ultraviolet divergencies in this problem was clarified in Ref. [10].

Undoubtedly, it is interesting to extend these theoretical studies to finite temperature. It is worth noting here that the total number of papers concerned with the calculation of the Casimir effect at finite temperature, and especially for spherical boundaries, is not so large. First Ref. [11] should be mentioned, where the multiple scattering expansion has been developed in order to investigate the vacuum effects for perfectly conducting boundaries. The calculation of the vacuum electromagnetic energy of a compact ball with the same velocity of light inside the ball and in the surrounding medium proves to be not more involved. In papers by Brevik and co-authors [3,12–16] this problem has been studied at zero temperature and at finite temperature, as well as with allowance for dispersion (see also Refs. [17–20]). However, the Casimir effect at finite temperature for a dielectric ball made of nonmagnetic material has not been considered till now.

An essential advantage of the calculation of the Casimir energy of a dilute dielectric ball, carried out in Ref. [9] by the mode summation method, is the possibility for its straightforward generalization to the finite temperature. It is this problem that will be considered in the present paper. The employment of the addition theorem for the Bessel functions enables one to carry out the summation over the angular momentum in a closed form. As a result, the exact (in the Δn^2 approximation) value for the Casimir internal and free energies of a dilute dielectric ball will be derived for finite temperature also. The divergencies, inevitable in such studies, will be removed by making use of the renormalization procedure developed previously for calculation of the relevant Casimir energy at zero temperature. Both thermodynamic characteristics are presented as the sum of the respective quantity for a compact ball with uniform velocity of light and an additional term which is specific only for a pure dielectric ball. The behavior of the thermodynamic characteristics in the low and high temperature limits is investigated. The low temperature expansions for the internal and free energy involve, except for the constant term, only even powers of the temperature T beginning from T^4 .

The layout of the paper is as follows. In Sec. II the general formulas are introduced for the internal and free energies of a system of noninteracting oscillators in terms of the sum over the Matsubara frequencies. This enables one to avoid the ambiguities arising when the

formal substitution of the integration over the imaginary frequencies by the summation over the discrete Matsubara frequencies is used in the integral representation for the relevant Casimir energy at zero temperature. The renormalization procedure needed for removing the divergencies is also discussed here. In Sec. III first the internal Casimir energy of a dilute dielectric ball is calculated. Next the free energy is obtained by partial integration of the relevant thermodynamic relation. The low and high temperature limits of the thermodynamic characteristics are examined. In the Conclusion (Sec. IV) the obtained results are summarized and future studies in this area are outlined.

II. TRANSITION TO THE FINITE TEMPERATURE IN CALCULATIONS OF THE CASIMIR ENERGY

Usually the transition to finite temperature in calculations of vacuum energy is accomplished by substituting the integration over imaginary frequencies by summation over the discrete Matsubara frequencies in the integral representation for the Casimir energy at zero temperature [16,21,22]. However, following this way one should control how many times the integration by parts in the integral expression at hand has been done [23,24]. The point is that in this way one may obtain both the internal energy and the free energy at finite temperature of the system under consideration. The corresponding examples can be found in Ref. [23].

Keeping this in mind, we shall proceed from the general formulas determining the internal energy and the free energy of a set of noninteracting oscillators at finite temperature. Here we briefly remind the readers of these formulas and their simple derivation. The natural system of units is used, where $c = \hbar = k_B = 1$, k_B being the Boltzman constant.

Let us consider an infinite set of noninteracting oscillators with frequencies determined by the equations

$$f_{\{p\}}(\omega, a) = 0. \quad (2.1)$$

Here a denotes some parameters specifying the configuration of the system at hand and $\{p\}$ is a complete set of quantum numbers characterizing the spectrum. In the case under consideration a stands for the radius of the ball, and $\{p\}$ incorporates the orbital (l) and azimuthal (m) quantum numbers and the type of the solutions of the relevant Maxwell equations [the transverse electric (TE) and transverse magnetic (TM) modes]. The free energy of such a set of noninteracting oscillators at finite temperature T is determined by the formula

$$F(T) = T \sum_{\{p\}} \sum_{n=0}^{\infty}{}' \ln f_{\{p\}}(i\omega_n, a), \quad (2.2)$$

where ω_n are the Matsubara frequencies

$$\omega_n = 2\pi nT, \quad (2.3)$$

and the prime on the summation sign means that the $n = 0$ term is counted with half weight.

Derivation of the formula (2.2) can be found in Ref. [23], where the free energies of individual oscillators

$$F_1(T, \omega) = \frac{\omega}{2} + T \ln(1 - e^{-\omega/T}) \quad (2.4)$$

have been summed by the contour integration method. In Eq. (2.4) ω are the roots of the frequency equations (2.1). It is assumed that for a given set $\{p\}$ Eq. (2.1) has an infinite countable sequence of the roots. In order to find the sum of the free energies (2.4) corresponding to this sequence of the roots, in paper [23] the contour integration has been used.

Having obtained the free energy (2.2) one can derive the internal energy $U(T)$ of the set of noninteracting oscillators by making use of the thermodynamic relation

$$U(T) = \frac{\partial}{\partial \beta} [\beta F(T)], \quad \beta = T^{-1}. \quad (2.5)$$

Substitution of Eq. (2.2) into this relation gives

$$U(T) = -T \sum_{\{p\}} \sum_{n=0}^{\infty} \omega_n \frac{d}{d\omega_n} \ln f_{\{p\}}(i\omega_n, a). \quad (2.6)$$

Formula (2.6) can be derived directly by summing the internal energies of the individual oscillators

$$U_1(T, \omega) = \frac{\omega}{2} \coth\left(\frac{\beta\omega}{2}\right) \quad (2.7)$$

and applying for this purpose the contour integration [20]. In this case the sum over the Matsubara frequencies (2.3) in Eq. (2.6) arises as a result of evaluation of the respective contour integral by the residue theorem, the residues being taken at the poles of the function $\coth(\beta\omega/2)$.

Certainly, the representations (2.2) and (2.6) are formal because they involve the divergencies. Therefore in order to obtain the physical results, an appropriate renormalization should be done. This procedure includes specifically the subtraction of the vacuum energy of unbounded homogeneous space [25,26]. This can be achieved by the following substitution in Eqs. (2.2) and (2.6)

$$f_{\{p\}}(i\omega_n, a) \rightarrow \frac{f_{\{p\}}(i\omega_n, a)}{f_{\{p\}}(i\omega_n, \infty)}. \quad (2.8)$$

In Ref. [9] it was shown that in the case of a dielectric ball an additional renormalization is to be done, namely, the contribution into the vacuum energy, which is proportional to Δn , should also be subtracted. Here $\Delta n = n_1 - n_2$, with n_1 (n_2) being the refractive index inside (outside) the ball. We shall follow this scheme at finite temperature too (see the reasoning in the next section).

III. INTERNAL AND FREE ENERGIES OF A DILUTE DIELECTRIC BALL AT FINITE TEMPERATURE

We shall consider a solid ball of radius a placed in an unbounded uniform medium, the temperature T of the ball and of the ambient medium being the same. The nonmagnetic materials making up the ball and its surroundings are characterized by permittivity ε_1 and ε_2 , respectively. It is assumed that the conductivity in both the media is zero. Further we put

$$\sqrt{\varepsilon_1} = n_1 = 1 + \frac{\Delta n}{2}, \quad \sqrt{\varepsilon_2} = n_2 = 1 - \frac{\Delta n}{2} \quad (3.1)$$

and assume that $\Delta n \ll 1$. From here it follows, in particular, that

$$\varepsilon_1 - \varepsilon_2 = (n_1 + n_2)(n_1 - n_2) = 2 \Delta n. \quad (3.2)$$

In the problem at hand, as well as in the other ones (see the examples in Ref. [23]), it is convenient to calculate first the internal energy of a dielectric ball using Eq. (2.6) and then to get the free energy by integrating the thermodynamic relation (2.5).

Equations, determining the frequencies of the electromagnetic oscillations associated with a dielectric ball, are [27]

$$\Delta_l^{\text{TE}}(a\omega) = 0, \quad \Delta_l^{\text{TM}}(a\omega) = 0, \quad l = 1, 2, \dots \quad (3.3)$$

For pure imaginary frequencies $\omega = i\omega_n$, with ω_n being the Matsubara frequencies (2.3), the left-hand sides of Eqs. (3.3) are given by

$$\begin{aligned} \Delta_l^{\text{TE}}(ia\omega_n) &= \sqrt{\varepsilon_1} s'_l(k_{1n}a) e_l(k_{2n}a) - \sqrt{\varepsilon_2} s_l(k_{1n}a) e'_l(k_{2n}a), \\ \Delta_l^{\text{TM}}(ia\omega_n) &= \sqrt{\varepsilon_2} s'_l(k_{1n}a) e_l(k_{2n}a) - \sqrt{\varepsilon_1} s_l(k_{1n}a) e'_l(k_{2n}a), \end{aligned} \quad (3.4)$$

where $k_{\alpha n} = \sqrt{\varepsilon_{\alpha}} \omega_n$, $\alpha = 1, 2$, and $s_l(x)$, $e_l(x)$ are the modified Riccati–Bessel functions [28]

$$s_l(x) = \sqrt{\frac{\pi x}{2}} I_{\nu}(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}} K_{\nu}(x), \quad \nu = l + \frac{1}{2}. \quad (3.5)$$

The prime in Eq. (3.4) stands for the differentiation with respect to the entire argument of the Riccati–Bessel functions. The permittivities ε_{α} , $\alpha = 1, 2$ are assumed to be independent of the frequency ω (dispersion is ignored) and of the temperature T .

Following the same way as in calculations of the Casimir energy at zero temperature [9] and making use of Eqs. (2.6), (2.8), and (3.4), we obtain the internal energy of a dielectric ball in the form

$$U(T) = -T \sum_{l=1}^{\infty} (2l+1) \sum_{n=0}^{\infty} w_n \frac{d}{dw_n} \ln \left[W_l^2(n_1 w_n, n_2 w_n) - \frac{\Delta n^2}{4} P_l^2(n_1 w_n, n_2 w_n) \right], \quad (3.6)$$

where

$$W_l(n_1 w_n, n_2 w_n) = s_l(n_1 w_n) e'_l(n_2 w_n) - s'_l(n_1 w_n) e_l(n_2 w_n), \quad (3.7)$$

$$P_l(n_1 w_n, n_2 w_n) = s_l(n_1 w_n) e'_l(n_2 w_n) + s'_l(n_1 w_n) e_l(n_2 w_n), \quad (3.8)$$

and we have introduced the dimensionless Matsubara frequencies

$$w_n = a\omega_n = 2\pi n a T, \quad n = 0, 1, 2, \dots \quad (3.9)$$

It is easy to be convinced that Eq. (3.6) can be derived from Eq. (2.12) in paper [9] by the substitution

$$dy \rightarrow 2\pi a T \sum_{n=0}^{\infty} \delta(y - w_n) dy. \quad (3.10)$$

Comparing Eq. (2.2) and Eq. (2.12) in Ref. [9] one arrives at the following inference. In order to get the free energy at finite temperature by making use of the substitution (3.10), one integration by parts should preliminary be done in Eq. (2.12) in [9]. Thus proceeding from the well justified equations for the free energy (2.2) and for the internal energy (2.6) at finite temperature, one can escape necessity to solve the problem: which energy (free or internal) is obtained on the substitution (3.10) in the initial integral representation for the Casimir energy at zero temperature [16,24].

In Eq. (3.6) we have subtracted the contribution of an unbounded homogeneous medium obtained in the limit $a \rightarrow \infty$. As in the case of zero temperature, it gives, specifically, the term linear in Δn [see Eq. (2.11) in Ref. [9]]. According to the renormalization procedure developed in Ref. [9] this contribution should be canceled by the corresponding counter term.

The necessity to subtract the contributions into the vacuum energy linear in $\varepsilon_1 - \varepsilon_2$ is justified by the following consideration. The Casimir energy of a dilute dielectric ball can be thought of as the net result of the van der Waals interactions between the molecules making up the ball [4]. These interactions are proportional to the dipole momenta of the molecules, i.e., to the quantity $(\varepsilon_1 - 1)^2$. Thus, when a dilute dielectric ball is placed in the vacuum, then its Casimir energy should be proportional to $(\varepsilon_1 - 1)^2$. It is natural to assume that when such a dielectric ball is surrounded by an infinite dielectric medium with permittivity ε_2 , then its Casimir energy should be proportional to $(\varepsilon_1 - \varepsilon_2)^2$. The physical content of the contribution into the vacuum energy linear in $\varepsilon_1 - \varepsilon_2$ has been investigated in the framework of the microscopic model of the dielectric media (see Ref. [29], and references therein). It has been shown that this term represents the self-energy of the electromagnetic field attached to the polarizable particles or, in more detail, it is just the sum of the individual atomic Lamb shifts. Certainly this term in the vacuum energy should be disregarded when calculating the Casimir energy originated in the electromagnetic interaction between different polarizable particles or atoms [5,7,30–32].

However, there is an opposite point of view on the Δn contribution to the vacuum energy of a pure dielectric ball according to which this term has a real physical meaning and when calculating its value an ultraviolet cutoff should be introduced. In this problem there is a natural cutoff. Really, if d is a typical distance between the atoms or molecules inside the ball then photons with energy grater than d^{-1} do not “feel” the dielectric body and propagate freely. This point of view goes back to the series of papers by Schwinger who has tried to explain in this way the sonoluminescence [33]. Further development of this approach

can be found in Refs. [34]. Controversy on this subject is going on (see, for example, Ref. [35]). In any case the Δn^2 contribution has, without doubts, real physical meaning and it is this term that is considered below.

For arbitrary material media inside and outside of the ball with permittivities ε_1 , ε_2 and permeabilities μ_1 , μ_2 , respectively, the following substitutions should be done in Eq. (3.6)

$$n_i \rightarrow \frac{1}{c_i} = \sqrt{\varepsilon_i \mu_i}, \quad i = 1, 2, \quad (3.11)$$

$$\frac{\Delta n^2}{4} \rightarrow \left(\frac{\sqrt{\varepsilon_1 \mu_2} - \sqrt{\varepsilon_2 \mu_1}}{\sqrt{\varepsilon_1 \mu_2} + \sqrt{\varepsilon_2 \mu_1}} \right)^2 \equiv \xi^2. \quad (3.12)$$

With account of these substitutions it easy to do the transition to continuous velocity of light on the surface of a compact ball placed in material surroundings $\varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 = c^{-2}$. In this case the internal Casimir energy is again determined by Eq. (3.6) with obvious changes

$$\begin{aligned} W_l \left(\frac{w_n}{c}, \frac{w_n}{c} \right) &= -1, \\ P_l \left(\frac{w_n}{c}, \frac{w_n}{c} \right) &= \left[s_l \left(\frac{w_n}{c} \right) e_l \left(\frac{w_n}{c} \right) \right]', \\ \xi^2 &= \left(\frac{\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}{\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}} \right)^2 = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right)^2 = \left(\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \right)^2. \end{aligned} \quad (3.13)$$

Now we return to consideration of a dilute dielectric ball and content ourselves with the Δn^2 approximation. In this case the contributions of W_l^2 and P_l^2 into the internal energy (3.6) are additive

$$U(T) = U_P(T) + U_W(T). \quad (3.14)$$

In the approximation chosen we can put $P_l(n_1 w_n, n_2 w_n) \simeq P_l(w_n, w_n)$ with the result

$$U_P(T) = \frac{\Delta n^2}{4} T \sum_{l=1}^{\infty} (2l+1) \sum_{n=0}^{\infty} w_n \frac{d}{dw_n} P_l^2(w_n, w_n). \quad (3.15)$$

Analysis of divergencies in the problem at hand, carried out in paper [9], leads to the following recipe for obtaining the contribution into the internal energy of the W_l^2 term in the argument of the logarithm function in Eq. (3.6). It is sufficient to calculate the contribution of the W_l^2 term alone and then to change the sign of this contribution to the opposite one. Hence

$$U_W(T) = T \sum_{l=1}^{\infty} (2l+1) \sum_{n=0}^{\infty} w_n \frac{d}{dw_n} W_l^2(n_1 w_n, n_2 w_n), \quad (3.16)$$

and only the term proportional to Δn^2 should be preserved in this expression.

By making use of the addition theorem for the Bessel functions [28] the sum over the angular momentum l in Eqs. (3.15) and (3.16) can be represented in a closed form in the same way as it has been done at zero temperature in papers [9,19]

$$\sum_{l=1}^{\infty} (2l+1) P_l^2(w_n, w_n) = \frac{1}{2} \int_0^2 \left[\frac{d}{dw_n} \left(\frac{w_n}{R} e^{-2w_n R} \right) \right]^2 R dR - e^{-4w_n}, \quad (3.17)$$

$$\sum_{l=1}^{\infty} (2l+1) W_l^2(n_1 w_n, n_2 w_n) = \frac{\Delta n^2}{8} \int_{\Delta n}^2 \frac{e^{-2w_n R}}{R^5} (4 + R^2 + 4w_n R - w_n R^3)^2 dR - e^{2\Delta n w_n}. \quad (3.18)$$

Upon substituting the expressions (3.17) and (3.18) into Eqs. (3.15) and (3.16), respectively, first the differentiation d/dw_n should be done, and only after that the integral over dR must be evaluated. It gives

$$U_P(T) = \frac{\Delta n^2}{4} \sum_{n=0}^{\infty} \left[\left(2w_n^2 + 2w_n + \frac{1}{2} \right) e^{-4w_n} - \frac{1}{2} \right], \quad (3.19)$$

$$\begin{aligned} U_W(T) = \frac{\Delta n^2}{8} T \sum_{n=0}^{\infty} \left\{ (1 + 4w_n) e^{-4w_n} \right. \\ \left. - \left[1 - 2\Delta n + \frac{16}{\Delta n} + \frac{w_n^2}{\Delta n^2} (16 - 8\Delta n^2 + \Delta n^4) \right] e^{-2\Delta n w_n} \right. \\ \left. + 16w_n^2 \int_{\Delta n}^2 \frac{e^{-2w_n R}}{R} dR \right\}. \end{aligned} \quad (3.20)$$

In Eq. (3.20) only the terms proportional to Δn^2 should be preserved [9], the rest of the terms being irrelevant to our consideration. When deriving Eq. (3.20) we have dropped the last term in Eq. (3.18), $e^{2\Delta n w_n}$, which gives rise to divergence when calculating the sum over the Matsubara frequencies.

First we consider the internal energy (3.19). Summation over the Matsubara frequencies can be done by making use of the formula

$$\sum_{n=0}^{\infty} e^{-4w_n} = \frac{1}{2} \coth(4\pi a T) = \frac{1}{2} + \frac{1}{e^{8\pi a T} - 1}. \quad (3.21)$$

It gives

$$U_P(T) = \frac{\Delta n^2}{8} T \left[t^2 \frac{\coth(2t)}{\sinh^2(2t)} + \frac{t}{\sinh^2(2t)} + \frac{1}{2} \coth(2t) \right], \quad t = 2\pi a T. \quad (3.22)$$

It is worth noting that the last term under the sum sign in Eq. (3.19) gives zero contribution, when the zeta regularization technique [36,37] is applied

$$-\frac{1}{2} \sum_{n=0}^{\infty} n^0 = -\frac{1}{2} \left(\frac{1}{2} + \zeta(0) \right) = 0, \quad (3.23)$$

where $\zeta(z)$ is the Riemann zeta function, $\zeta(0) = -1/2$. At zero temperature this term gives rise to a divergence that has been removed by respective subtraction [9].

From Eq. (3.22) we deduce the following behaviour of the internal energy $U_P(T)$ at low temperature

$$U_P(T) = \frac{5\Delta n^2}{128\pi a} + \frac{2}{45} \Delta n^2 (\pi a)^3 T^4 + \frac{128}{945} \Delta n^2 (\pi a)^5 T^6 - \frac{512}{525} \Delta n^2 (\pi a)^7 T^8 + \mathcal{O}(T^{10}). \quad (3.24)$$

Integration of the thermodynamic relation (2.5) enables one to get the free energy

$$F(T) = -T \int \frac{U(T)}{T^2} dT + C T, \quad (3.25)$$

where C is a constant. Upon substitution of Eq. (3.22) into Eq. (3.25) the first two terms can be integrated explicitly [20], the last term in Eq. (3.22) leads to the integral

$$\int \frac{dx}{x} \coth(x),$$

which cannot be expressed in terms of the table integrals [38]. Further we shall use Eq. (3.25) for obtaining the asymptotics of the free energy, keeping in mind that the internal energy $U_W(T)$ in Eq. (3.20) cannot be represented in a simple closed form such as Eq. (3.22).

Substituting the asymptotics (3.24) into Eq. (3.25) we obtain the respective free energy in the low temperature region

$$F_P(T) = \frac{5\Delta n^2}{128\pi a} - \frac{2}{135}\Delta n^2(\pi a)^3 T^4 - \frac{128}{4725}\Delta n^2(\pi a)^2 T^6 + \frac{512}{3675}\Delta n^2(\pi a)^7 T^8 + \mathcal{O}(T^{10}). \quad (3.26)$$

Here the linear in T term CT has been dropped, because the requirement that the entropy $S_P(T)$ should vanish at $T = 0$ gives [20]

$$S_P(0) = \lim_{T \rightarrow 0} T^{-1} (U_P(T) - F_P(T)) = C = 0. \quad (3.27)$$

Hence, at low temperature the expansions both for the internal energy (3.24) and for the free energy (3.26) involve only even powers of the temperature beginning from T^4 . At zero temperature we have

$$U_P(0) = F_P(0) = E_P = \frac{5\Delta n^2}{128\pi a}, \quad (3.28)$$

where E_P is the Casimir energy of a compact ball with the same velocity of light inside and outside the ball [9,19,11].

Our calculation of the free energy $F_P(T)$ corresponds to the two-scattering approximation in treatment of a perfectly conducting spherical shell in Ref. [11]. The relevant results of that paper should be multiplied by $\Delta n^2/4$ before comparing with ours. However, the free energy of a conducting sphere, calculated in the two-scattering approximation, is presented there only graphically, and Eq. (8.37) in that paper gives the low temperature behavior of an exact result for this quantity. Therefore the coefficients in this equation are a bit different as compared with the two first terms in our Eq. (3.26).

When temperature T tends to infinity, Eq. (3.22) gives

$$U_P(T) \simeq \frac{\Delta n^2}{16} T, \quad T \rightarrow \infty. \quad (3.29)$$

Substituting this asymptotics into Eq. (3.25) we arrive at the high temperature limit for the free energy $F_P(T)$

$$F_P(T) \simeq -\frac{\Delta n^2}{16} T [\ln(aT) + \alpha], \quad T \rightarrow \infty. \quad (3.30)$$

The constant α can be found by making use of the complete expression for $F_P(T)$ (see Refs. [20,39])

$$\alpha = \gamma + \ln 4 - \frac{5}{4}.$$

We have explained above how to do the transition to continuous velocity of light on the surface of a material ball [see Eqs. (3.11)–(3.13)]. With allowance for this we immediately conclude that the internal energy $U_P(T)$ and free energy $F_P(T)$ exactly concern that configuration, certainly upon the substitution (3.12). Our functions $U_P(T)$ and $F_P(T)$ completely coincide with calculations in Refs. [19,20] where the addition theorem for the Bessel functions has been applied also. But in our problem (a pure dielectric ball) there is an additional contribution to the vacuum energy generated by the functions W_l^2 in Eq. (3.6). Now we turn to the analysis of this contribution.

The summation over the Matsubara frequencies in Eq. (3.20) gives

$$\begin{aligned} U_W(T) = \frac{\Delta n^2}{8} T \left\{ \frac{1}{e^{4t} - 1} \left(1 + \frac{4t}{1 - e^{-4t}} \right) - \frac{1}{e^{2\Delta n t} - 1} \right. \\ \left. - \frac{t}{2 \sinh^2(\Delta n t)} \left[\frac{8}{\Delta n} - \Delta n - 2t \left(2 - \frac{4}{\Delta n^2} - \frac{\Delta n^2}{4} \right) \coth(\Delta n t) \right] \right. \\ \left. + t^2 \int_{\Delta n}^2 \frac{dR}{R} \frac{\coth(tR)}{\sinh^2(tR)} \right\}, \quad t = 2\pi a T. \end{aligned} \quad (3.31)$$

According to the renormalization procedure employed, only the terms proportional to Δn^2 should be retained in Eq. (3.31). Obviously, this can be accomplished explicitly when considering the asymptotics of the internal energy $U_W(T)$ for low and high temperatures. For low T Eq. (3.31) gives

$$\begin{aligned} U_W(T) = \frac{\Delta n^2}{48\pi a} - \frac{16}{45} \Delta n^2 (\pi a)^2 T^4 + \frac{1024}{2835} \Delta n^2 (\pi a)^5 T^6 \\ - \frac{4096}{7875} \Delta n^2 (\pi a)^7 T^8 + \mathcal{O}(T^{10}). \end{aligned} \quad (3.32)$$

By making use of Eq. (3.25) with the constant C equal to zero, we obtain the low temperature expansion for the respective free energy

$$\begin{aligned} F_W(T) = \frac{\Delta n^2}{48\pi a} + \frac{16}{135} \Delta n^2 (\pi a)^3 T^4 - \frac{1024}{14175} \Delta n^2 (\pi a)^5 T^6 \\ + \frac{4096}{55125} \Delta n^2 (\pi a)^7 T^8 + \mathcal{O}(T^{10}). \end{aligned} \quad (3.33)$$

The sum of Eqs. (3.24) and (3.32) gives the total internal energy at low temperature

$$\begin{aligned} U(T) = \frac{23}{384} \frac{\Delta n^2}{\pi a} - \frac{14}{45} \Delta n^2 (\pi a)^3 T^4 + \frac{1408}{2835} \Delta n^2 (\pi a)^5 T^6 \\ - \frac{11776}{7875} \Delta n^2 (\pi a)^7 T^8 + \mathcal{O}(T^{10}). \end{aligned} \quad (3.34)$$

From Eqs. (3.26) and (3.33) we obtain for the total free energy of a dilute dielectric ball

$$F(T) = \frac{23}{384} \frac{\Delta n^2}{\pi a} + \frac{14}{135} \Delta n^2 (\pi a)^3 T^4 - \frac{1408}{14175} \Delta n^2 (\pi a)^5 T^6 + \frac{11776}{55125} \Delta n^2 (\pi a)^7 T^8 + \mathcal{O}(T^{10}). \quad (3.35)$$

The first three terms of the asymptotics (3.35) has been derived in a recent paper by Barton [40] in the framework of a completely different approach, namely by making use of perturbative theory for quantized electromagnetic field where dielectric ball is considered as a perturbation in unbounded continuous surroundings. When comparing our Eq. (3.35) with respective formula in the Barton paper [40] one should take into account that our quantity Δn^2 is equal to the Barton's $4\pi^2(n\alpha)^2$. In Ref. [40] an additional term proportional to T^3 has been obtained for the free energy in the low temperature limit. It should be noted that the T^3 term does not give contribution to the Casimir pressure exerted on the surface of a dielectric ball. In this respect this term is nonobservable. In our approach the T^3 term is absent because at first we have calculated the total energy $U(T)$ and after that we derive the free energy $F(T)$ proceeding from $U(T)$. At zero temperature we have

$$U(0) = F(0) = E = \frac{23}{384} \frac{\Delta n^2}{\pi a}, \quad (3.36)$$

where E is the Casimir energy of a dilute dielectric ball calculated in Ref. [9].

When passing from a compact ball with uniform velocity of light to a pure dielectric ball, the sign of the first temperature correction ($\sim T^4$) to the free energy and consequently to the Casimir forces changes to the opposite one (see Eqs. (3.26) and (3.35)). Keeping two terms in the expansion (3.35) we get for the Casimir forces exerted on the unit area of the ball surface

$$\mathcal{F} = -\frac{1}{4\pi a^2} \frac{\partial F(T)}{\partial a} = \frac{23}{1636} \frac{\Delta n^2}{\pi^2 a^4} \left[1 - \frac{112}{345} (2\pi a T)^4 \right]. \quad (3.37)$$

From Eq. (3.20) one can derive the high temperature behaviour of the internal energy $U_W(T)$ in the same way as Eq. (3.29) has been obtained

$$U_W(T) \simeq \frac{\Delta n^2}{8} T \frac{1}{2} = \frac{\Delta n^2}{16} T, \quad T \rightarrow \infty. \quad (3.38)$$

Substitution of this result into Eq. (3.25) and subsequent integration gives

$$F_W \simeq -\frac{\Delta n^2}{16} T [\ln(aT) + \beta], \quad (3.39)$$

where β is a constant.

By making use Eqs. (3.29), (3.30), (3.38), and (3.39) we obtain the high temperature asymptotics of the internal energy and free energy of a dilute dielectric ball

$$U(T) = U_P(T) + U_W(T) \simeq \frac{\Delta n^2}{8} T, \quad T \rightarrow \infty, \quad (3.40)$$

$$F(T) = F_P + F_W(T) \simeq -\frac{\Delta n^2}{8} T [\ln(aT) + c], \quad T \rightarrow \infty, \quad (3.41)$$

where $c = \alpha + \beta$ is a constant the exact value of which can be obtained by other methods [40,39]

$$c = \ln 4 + \gamma - \frac{7}{8}, \quad \beta = \frac{3}{8}. \quad (3.42)$$

Exactly the same high temperature behaviour of the thermodynamic functions of electromagnetic field connected with a dilute dielectric ball has been obtained in Ref. [40].

With allowance for the dimension of the respective quantities it is easy to be convinced that Eq. (3.40) and the last term in Eq. (3.41) do not contain the Planck constant. Hence, the high temperature limit for the internal and free energies implies the classical limit [41,42]. At the same time these leading terms do not depend on the radius of the ball a too and, as a result, they do not contribute to the Casimir force at high temperature. From the asymptotics (3.34), (3.35), (3.40), and (3.41) it follows that the thermodynamic characteristics $U(T)$ and $F(T)$ of a dilute dielectric ball have, respectively, minimum and maximum at nonzero values of the temperature T .

The characteristic temperature scale for the thermodynamic quantities under consideration is determined by the radius a of the ball. For $a \sim 10^{-4}$ cm this scale proves to be large ~ 1000 °K.

IV. CONCLUSION

In this paper the Casimir internal and free energies are calculated for a dilute dielectric ball at finite temperature. As we are aware, it has been done for the first time. The explicit formulas are derived which allow one to develop the expansions for thermodynamic characteristics of the ball at low and high temperatures. It is found that the first temperature correction ($\sim T^4$) to the free energy in the problem at hand has an opposite sign as compared with a perfectly conducting sphere [11] and a compact ball with constant velocity of light inside the ball and in the surroundings [16,20]. It implies that the Casimir force, exerted on the surface of a dielectric ball and tending to expand it, diminishes with rising temperature [see Eq. (3.37)]. Usually the temperature dependence of the Casimir forces is opposite [26,41,42]. However, for a perfectly conducting wedge in the low temperature region the Casimir forces also decrease when the temperature rises [11].

Without doubt, it is worth considering this problem in the framework of other methods, for example, by making use of the Green's function techniques. However, before doing this the role of the so-called contact terms in the expression of the vacuum energy employed there should be elucidated. In our global approach these terms do not appear because we are only dealing with the sum of the eigenfrequencies of the electromagnetic field under given boundary conditions.

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REFERENCES

- [1] K. A. Milton, Ann. Phys. (N.Y.) **127**, 49 (1980).
- [2] K. A. Milton and Y. J. Ng, Phys. Rev. E **55**, 4207 (1997).
- [3] I. Brevik, V. V. Nesterenko, and I. G. Pirozhenko, J. Phys. A **31**, 8661 (1998).
- [4] K. A. Milton and Y. J. Ng, Phys. Rev. E **57**, 5504 (1998).
- [5] G. Barton, J. Phys. A **32**, 525 (1999).
- [6] I. Brevik, V. N. Marachevsky, Phys. Rev. D **60**, 085006 (1999).
- [7] I. Brevik, V. N. Marachevsky, and K. A. Milton, Phys. Rev. Lett. **82**, 3948 (1999).
- [8] V. N. Marachevsky, “Casimir-Polder energy and dilute dielectric ball: nondispersive case,” Report No. SPb-IP-99-11, hep-th/9909210; “Casimir energy and realistic model of dilute dielectric ball,” hep-th/0101062.
- [9] G. Lambiase, G. Scarpetta, and V. V. Nesterenko, “Casimir energy of a dilute dielectric ball in the mode summation method,” hep-th/9912176.
- [10] M. Bordag, K. Kirsten, and D. Vassilevich, Phys. Rev. D **59**, 085011 (1999).
- [11] R. Balian and B. D. Duplantier, Ann. Phys. (N.Y.) **104**, 300 (1977); **112**, 165 (1978).
- [12] I. Brevik and H. Kolbenstvedt, Ann. Phys. (N.Y.) **143**, 179 (1982); **149**, 237 (1983); Can. J. Phys. **62**, 805 (1984); **63**, 1409 (1985).
- [13] I. Brevik and G. Einevoll, Phys. Rev. D **37**, 2977 (1988).
- [14] I. Brevik and I. Clausen, Phys. Rev. D **39**, 603 (1989).
- [15] I. Brevik, J. Phys. A **20**, 5189 (1987).
- [16] I. Brevik and T. A. Yousef, J. Phys. A **33**, 5819 (2000).
- [17] V. V. Nesterenko and I. G. Pirozhenko, JETP Lett. **67**, 445 (1998).
- [18] G. Lambiase, V. V. Nesterenko, and M. Bordag, J. Math. Phys. **40**, 6254 (1999).
- [19] I. Klich, Phys. Rev. D **61**, 025004 (2000); and (to be published,) hep-th/0012241.
- [20] I. Klich, J. Feinberg, A. Mann, and M. Revzen, Phys. Rev. D **62**, 045017 (2000).
- [21] M. H. Bernstein, I. Brevik, and S. D. Odintsov, Ann. Phys. (N.Y.) **257**, 84 (1997).
- [22] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Mechanics*, (Pergamon, Oxford, 1984), Pt. 2.
- [23] V. V. Nesterenko and I. G. Pirozhenko, J. Math. Phys. **38**, 6265 (1997).
- [24] I. Klich (private communication).
- [25] G. Plunien, B. Muller, W. Greiner, Phys. Rep. **134**, 87 (1986).
- [26] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and its Applications* (Clarendon Press, Oxford, 1997).
- [27] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
- [28] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1972).
- [29] P. W. Milonni, M. Schaden, and L. Spruch, Phys. Rev. A **59**, 4259 (1999).
- [30] G. Barton, “Perturbative Casimir energies of dispersive spheres, cubes, and cylinders,” Preprint University of report, Brighton, England, 2000.
- [31] J. S. Høye and I. Brevik, J. Stat. Phys. **100**, 223 (2000).
- [32] J. S. Høye, I. Brevik, and J. B. Aarseth, Phys. Rev. E **63**, 051101 (2001).
- [33] J. Schwinger, Proc. Nat. Acad. Sci. U.S.A. **90**, 958, **90**, 2105 (1993); **90**, 4505 (1993); **90**, 7285 (1993); **91**, 6473 (1994).
- [34] C.E. Carlson, C. Molina-París, J. Pérez-Mercader, Matt Visser, Phys. Lett. B **395**, 76

- (1997); Phys. Rev. D **56**, 1262 (1997); C. Molina-Paris and Matt Visser, *ibid.* **56**, 6629 (1997); S. Liberati, F. Belgiorno, M. Visser, and D.W. Sciama, J. Phys. A. **33**, 2251 (2000).
- [35] K. A. Milton, hep-th/0009173; S. Liberati, F. Belgiorno, and M. Visser, hep-th/0010140.
 - [36] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Technique with Applications* (World Scientific, Singapore, 1994).
 - [37] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions* (Springer, Berlin, 1995).
 - [38] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 5th ed. (Academic Press, New York, 1994).
 - [39] M. Bordag, V. V. Nesterenko, and I. G. Pirozhenko, “On the high temperature asymptotics of the free energy of quantum fields in confined regions,” hep-th/0107024.
 - [40] G. Barton, “Perturbative Casimir shifts of nondispersive spheres at finite temperature,” University of Sussex report, 2001.
 - [41] Kimball A. Milton, In *Applied Field Theory, Proc. 17th Symposium on Theoretical Physics*, edited by C. Lee, H. Min, and Q.-H. Park (Chungburn, Seoul, 1999); see also Report No. OKHEP-99-01, hep-th/9901011.
 - [42] J. Feinberg, A. Mann, and M. Revzen, Ann. Phys. (N.Y.) **288**, 103 (2001).